Linear Differentation Equation and Separable Equation Examples

1. Solve $ty' + (t+1)y = t$, $y(\log 2) = 1$

Soln:

$$y' + \frac{(t+1)y}{t} = 1$$

$$\mu y' + \frac{\mu(t+1)y}{t} = \mu$$

$$\cancel{\mu y'} + \frac{\mu(t+1)y}{t} = (\mu y)'$$

$$= \mu' y + \cancel{\mu y'}$$

$$\frac{\mu(t+1)\cancel{y}}{t} = \mu'\cancel{y}$$

$$\frac{t+1}{t} = \frac{\mu'}{\mu}$$

$$= (\ln(\mu))'$$

$$\int \frac{t+1}{t}\,dt = \ln(\mu)$$

$$\int \frac{t}{t}\,dt + \int \frac{1}{t}\,dt = \ln(\mu)$$

$$t + \ln(t) + C = \ln(\mu)$$

$$e^{t + \ln(t) + C} = \mu$$

let $c' = e^c$

$e^t \cdot t \cdot c' = \mu$

let $c' = 1$.

$\mu = e^t \cdot t$

$(\mu y)' = \mu$

$(\mu y) = \int \mu \, dt$

$(e^t \cdot t) y = \int e^t \cdot t \, dt$

We will solve $\int e^t \cdot t \, dt$ using integration by parts.

let $u = t$

let $v = e^t$

$u \int v \, dt - (\int u' (\int v \, dt) \, dt)$

$= t \int e^t \, dt - \int (t)' (\int e^t \, dt) \, dt$

$= te^t - \int 1 \cdot e^t \, dt$

$= te^t - e^t + c_1$

$(e^t \cdot t) y = e^t \cdot t - e^t + c_1$

$$y = \frac{e^t \cdot t - e^t + c_1}{e^t \cdot t}$$

Plug in $\log(2)$ for t and 1 for y.

$$1 = \frac{e^{\log 2} \cdot \log 2 - e^{\log 2} + C_1}{e^{\log 2} \cdot \log 2}$$

$$= \frac{2\log 2 - 2 + C_1}{2 \log 2}$$

$$2\log 2 = 2\log 2 - 2 + C_1$$

$$2 = C_1 \longrightarrow y = \frac{e^t \cdot t - e^t + 2}{e^t \cdot t}$$

2. Solve $(\sin t)y' + (\cos t)y = e^t$, $y(1) = a$

Soln:

$$y' + \left(\frac{\cos t}{\sin t}\right)y = \frac{e^t}{\sin t}$$

$$\mu y' + \left(\frac{\cos t}{\sin t}\right)\mu y = \frac{e^t \mu}{\sin t}$$

$$\cancel{\mu y'} + \left(\frac{\cos t}{\sin t}\right)\mu y = (\mu y)'$$

$$= \mu' y + \cancel{\mu y'}$$

$$\left(\frac{\cos t}{\sin t}\right)\mu \cancel{y} = \mu' \cancel{y}$$

$$\frac{\cos t}{\sin t} = \frac{\mu'}{\mu}$$

$$= (\ln(\mu))'$$

$$\int \frac{\cos t}{\sin t}\,dt = \ln(\mu)$$

let $u = \sin t$

$$\frac{du}{dt} = \cos t$$

$$dt = \frac{du}{\cos t}$$

$$\int \frac{1}{u}\, du$$

$= \ln|u| + c$

$= \ln|\sin t| + c$

$\ln(\mu) = \ln|\sin t| + c$

$\mu = \sin t \cdot e^c$

let $c' = e^c$

$\mu = \sin t \cdot c'$

let $c' = 1$

$\mu = \sin t$

$$(\mu y)' = \frac{e^t \mu}{\sin t}$$

$$((\sin t)\cdot y)' = \frac{e^t(\sin t)}{\sin t}$$

$= e^t$

$(\sin t)y = \int e^t\, dt$

$= e^t + C_1$

$$y = \frac{e^t + C_1}{\sin t}$$

Plug 1 for t and a for y.

$$a = \frac{e^1 + c_1}{\sin 1}$$

$$c_1 = a \cdot \sin 1 - e \longrightarrow y = \frac{e^t + a \cdot \sin 1 - e}{\sin t}$$

3. Solve $y' = \frac{2x}{1+2y}$, $y(2) = 0$

Soln:

$$\frac{dy}{dx} = \frac{2x}{1+2y}$$

$$(1+2y)\,dy = 2x\,dx$$

$$\int 1+2y\,dy = \int 2x\,dx$$

$$y + y^2 = x^2 + c$$

Plug in 2 for x and 0 for y.

$$0 = (2)^2 + c$$

$$c = -4 \longrightarrow y + y^2 = x^2 - 4$$

4. Solve $y' = \frac{x^2 + xy + y^2}{x^2}$

Soln:

Note: This is a homogenous equation.
A first order differential equation is homogeneous if it can be written in this form: $y' = f(y/x)$.

Going back to our equation, we see that
$$\frac{x^2+xy+y^2}{x^2} = 1 + \frac{y}{x} + \left(\frac{y}{x}\right)^2.$$
Since it can be written as $f(\frac{y}{x})$, it is a homogeneous equation.

We can solve this using separable equations, but first, we create a new variable $v = \frac{y}{x}$.

Since $v = \frac{y}{x}$, $y = vx$

$$\begin{aligned} y' &= (vx)' \\ &= v'x + vx' \\ &= v + v'x \quad (x'=1) \end{aligned}$$

Now, we have
$$v + v'x = 1 + v + v^2$$
$$v'x = 1 + v^2$$
$$\frac{dv}{dx}x = 1 + v^2$$
$$dv\, x = (1+v^2)\,dx$$
$$\frac{1}{1+v^2}\,dv = \frac{1}{x}\,dx$$

$$\int \frac{1}{1+v^2}\,dv = \int \frac{1}{x}\,dx$$

$\arctan(v) = \ln|x| + c$

$\arctan(v) - \ln|x| = c \longrightarrow$ Replace v with $\frac{y}{x}$.

$\arctan\left(\frac{y}{x}\right) - \ln|x| = c$

5. Solve $y' = \dfrac{x^2+y^2}{xy}$

Soln:

$$\frac{x^2+y^2}{xy} = \frac{x^2}{xy} + \frac{y^2}{xy}$$

$$= \frac{x}{y} + \frac{y}{x}$$

$$= \left(\frac{y}{x}\right)^{-1} + \frac{y}{x} \longleftarrow \text{Homogeneous equation}$$

Let $v = \dfrac{y}{x}$.

This means that $y = vx$.

$y' = (vx)'$

$= v'x + vx'$

$= v'x + v$

$v'x + v = v^{-1} + v$

$v'x = v^{-1}$

$$x\frac{dv}{dx} = \frac{1}{v}$$

$$v\,dv = \frac{1}{x}\,dx$$

$$\int v\,dv = \int \frac{1}{x}\,dx \longrightarrow \frac{v^2}{2} = \ln|x| + c$$

$$\frac{v^2}{2} - \ln|x| = c$$

Replace v with $\frac{y}{x}$. $\longrightarrow \dfrac{\left(\frac{y}{x}\right)^2}{2} - \ln|x| = c$